DYNAMICS OF SEMIGROUPS OF ENTIRE MAPS OF $\mathbb{C}^k$

SAYANI BERA AND RATNA PAL[1]

ABSTRACT. The goal of this paper is to study some basic properties of the Fatou and Julia sets for a family of holomorphic endomorphisms of $\mathbb{C}^k,\ k \geq 2$. We are particularly interested in studying these sets for semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k,\ k \geq 2$. We prove that if the Julia set of a semigroup G which is generated by endomorphisms of maximal generic rank k in $\mathbb{C}^k$ contains an isolated point, then G must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$. This generalizes a theorem of Fornaess–Sibony. Secondly, we define recurrent domains for semigroups and provide a description of such domains under some conditions.

1. INTRODUCTION

The purpose of this note is to study the Fatou–Julia dichotomy, not for the iterates of a single holomorphic endomorphism of $\mathbb{C}^k,\ k \geq 2$, but for a family $\mathcal{F}$ of such maps. The Fatou set of $\mathcal{F}$ will be by definition the largest open set where the family is normal, i.e., given any sequence in $\mathcal{F}$ there exists a subsequence which is uniformly convergent or divergent on all compact subsets of the Fatou set, while the Julia set of $\mathcal{F}$ will be its complement.

We are particularly interested in studying the dynamics of families that are semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k,\ k \geq 2$. For a collection $\{\psi_\alpha\}$ of such maps let

$$G = \langle \psi_\alpha \rangle$$

denote the semigroup generated by them. The index set to which α belongs is allowed to be uncountably infinite in general. The Fatou set and Julia set of this semigroup G will be henceforth denoted by $F(G)$ and $J(G)$ respectively. Also for a holomorphic endomorphism ϕ of $\mathbb{C}^k$, $F(\phi)$ and $J(\phi)$, will denote the Fatou set and Julia set for the family of iterations of ϕ. The ψ_α's that will be considered in the sequel will belong to one of the following classes:

- $\mathcal{E}_k$: The set of holomorphic endomorphisms of $\mathbb{C}^k$ which have maximal generic rank k.
- $\mathcal{I}_k$: The set of injective holomorphic endomorphisms of $\mathbb{C}^k$.
- $\mathcal{V}_k$: The set of volume preserving biholomorphisms of $\mathbb{C}^k$.
- $\mathcal{P}_k$: The set of proper holomorphic endomorphisms of $\mathbb{C}^k$.

The main motivation for studying the dynamics of semigroups in higher dimensions comes from the results of Hinkkanen–Martin[11] and Fornaess–Sibony [8]. While [11] considers the dynamics of semigroups generated by rational functions on the Riemann sphere, [8] puts forth several basic results about the dynamics of the iterates of a single holomorphic endomorphism of $\mathbb{C}^k,\ k \geq 2$. Under such circumstances, it seemed natural to us to study the dynamics of semigroups in higher dimensions.

Section 2 deals with basic properties of $F(G)$ and $J(G)$ when G is generated by elements that belong to $\mathcal{E}_k$ and $\mathcal{P}_k$. The main theorem in Section 3 states that if $J(G)$ contains an isolated point, then G must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$. Finally we define recurrent domains for semigroups in Section 4 and provide a classification of

1991 *Mathematics Subject Classification.* Primary: 32H02 ; Secondary : 32H50.
[1] Supported by CSIR-UGC(India) fellowship.

such domains under some conditions which are generalizations of the corresponding statements of Fornaess–Sibony [8] for the iterates of a single holomorphic endomorphism of $\mathbb{C}^k,\ k \geq 2$. The classification for recurrent Fatou components for the iterates of holomorphic endomorphisms of $\mathbb{P}^2$ and $\mathbb{P}^k$ is studied in [9] and [7] respectively. In [9] Fornaess–Sibony also gave a classification of recurrent Fatou components for iterations of Hénon maps inside K^+, which was initially considered by Bedford–Smillie in [4]. A classification for non-recurrent, non-wandering Fatou components of $\mathbb{P}^2$ is given in [10], whereas a classificiation of invariant Fatou components for nearly dissipative Hénon maps is studied in [5].

Acknowledgement: We would like to thank Kaushal Verma for valuable discussions and comments.

2. Properties of the Fatou set and Julia set for a semigroup G

In this section we will prove some basic properties of the Fatou set and the Julia set for semigroups.

Proposition 2.1. *Let G be a semigroup generated by elements of $\mathcal{E}_k$ where $k \geq 2$ and for any $\phi \in G$ define*

$$\Sigma_\phi = \{z \in \mathbb{C}^k : \det \phi(z) = 0\}.$$

Then for every $\phi \in G$

(i) $\phi(F(G) \setminus \Sigma_\phi) \subset F(G)$.
(ii) $J(G) \cap \phi(\mathbb{C}^k) \subset \phi(J(G))$*, if G is generated by elements of $\mathcal{P}_k$ or $\mathcal{I}_k$.*

Proof. Note that $\phi \in G$ is an open map at any point $z \in F(G) \setminus \Sigma_\phi$. Since for any sequence $\psi_n \in G$, the sequence $\psi_n \circ \phi$ has a convergent subsequence around a neighbourhood of z (say V_z), ψ_n also has a convergent subsequence on the open set $\phi(V_z)$ containing $\phi(z)$.
Now if G is generated by elements of $\mathcal{P}_k$ or $\mathcal{I}_k$ then ϕ is an open map at every point in $\mathbb{C}^k$. Then the Fatou set is forward invariant and hence the Julia set is backward invariant in the range of ϕ. ☐

A family of endomorphisms $\mathcal{F}$ in $\mathbb{C}^k$ is said to be locally uniformly bounded on an open set $\Omega \subset \mathbb{C}^k$ if for every point there exists a small enough neighbourhood of the point (say $V \subset \Omega$) such that $\mathcal{F}$ restricted to V is bounded i.e.,

$$\|f\|_V = \sup_V |f(z)| < M$$

for some $M > 0$ and for every $f \in \mathcal{F}$.

Proposition 2.2. *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$, where each $\phi_j \in \mathcal{E}_k$ and let Ω_G be a Fatou component of G such that G is locally uniformly bounded on Ω_G. Then for every $\phi \in G$ the image of Ω_G under ϕ i.e., $\phi(\Omega_G)$ is contained in Fatou set of G.*

Proof. Let $K \subset\subset \Omega_G$, i.e., K is a relatively compact subset of Ω_G, then

Claim:- Ω_G is a Runge domain i.e., $\hat{K} \subset \Omega_G$ where

$$\hat{K} := \{z \in \mathbb{C}^k : |P(z)| \leq \sup_K |P| \text{ for every polynomial } P\}.$$

Let $K_\delta = \{z \in \mathbb{C}^k :\ \operatorname{dist}(z, K) \leq \delta\}$. Choose $\delta > 0$ such that $K_\delta \subset\subset \Omega_G$. Now note that $\hat{K_\delta} \subset\subset \mathbb{C}^k$, $\hat{K_\delta} \supset \hat{K}$ and G is uniformly bounded on K_δ. Pick $\phi \in G$. Then there exists a polynomial endomorphism P_ϕ of $\mathbb{C}^k$ such that

$$|\phi(z) - P_\phi(z)| \leq \epsilon \text{ for every } z \in \hat{K_\delta}$$
$$\text{i.e.,} \qquad |P_\phi(z)| - \epsilon \leq |\phi(z)| \leq |P_\phi(z)| + \epsilon.$$

Hence

$$\begin{aligned}|\phi(z)| &\leq |P_\phi(z)| + \epsilon \leq \sup_{K_\delta} |P_\phi(z)| + \epsilon \\ &\leq \sup_{K_\delta} |\phi(z)| + 2\epsilon \leq M + 2\epsilon\end{aligned}$$

for every $z \in \hat{K_\delta}$ and some constant $M > 0$. So G is uniformly bounded on $\hat{K_\delta}$ and $\hat{K} \subset \Omega_G$. Let

$$\Sigma_i = \{z \in \mathbb{C}^k : \det \phi_i(z) = 0\}$$

for every $1 \leq i \leq n$ and

$$\Sigma = \bigcup_{i=1}^{n} \Sigma_i.$$

Thus ϕ_i for every i, where $1 \leq i \leq n$ is an open map in $\Omega_G \setminus \Sigma$. Hence $\phi_i(\Omega_G \setminus \Sigma)$ is contained inside a Fatou component say Ω_i and G is locally uniformly bounded on each of Ω_i for every $1 \leq i \leq n$ i.e., each Ω_i is a Runge domain.

Now pick $p \in \Omega_G \cap \Sigma$. Since Σ is a set with empty interior, there exists a sufficiently small disc centered at p say Δ_p such that $\overline{\Delta}_p \setminus \{p\} \subset \Omega_G \setminus \Sigma$. Then $\phi_i(\overline{\Delta}_p \setminus \{p\}) \subset \Omega_i$ for every $1 \leq i \leq n$ and since each Ω_i is Runge $\phi_i(p) \in \Omega_i$ i.e., $\phi_i(\Omega_G)$ is contained in the Fatou set for every $1 \leq i \leq n$. Now for any $\phi \in G$ there exists a $m > 0$ such that

$$\phi = \phi_{n_1} \circ \phi_{n_2} \circ \ldots \circ \phi_{n_m}$$

where $1 \leq n_j \leq n$ for every $1 \leq j \leq m$. Thus applying the above argument repeatedly for each $\phi_{n_j}(\tilde{\Omega}_j)$ where G is locally uniformly bounded on $\tilde{\Omega}_j$ it follows that $\phi(\Omega_G)$ is contained in the Fatou set of G. □

Proposition 2.3. *If $G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq n$ and let Ω_G be a Fatou component of G. Then for any $\phi \in G$ there exists a Fatou component of G, say Ω_ϕ such that $\phi(\Omega_G) \subset \bar{\Omega}_\phi$ and*

$$\partial\Omega_G \subset \bigcup_{i=1}^{n} \phi_i^{-1}(\partial\Omega_{\phi_i}).$$

Proof. Let $\phi \in G$ and let Σ_ϕ denote the set of points in $\mathbb{C}^k$ where the Jacobian of ϕ vanishes. Since $\Omega_G \setminus \Sigma_\phi$ is connected it follows that $\phi(\Omega_G \setminus \Sigma_\phi) \subset \Omega_\phi$ where Ω_ϕ is a Fatou component of G and by continuity $\phi(\Omega_G) \subset \bar{\Omega}_\phi$.

Pick $p \in \partial\Omega_G$ such that $p \notin \partial\Omega_{\phi_i}$ for every $1 \leq i \leq n$. Since $\phi_i(\Omega_G) \subset \bar{\Omega}_{\phi_i}$, $\phi_i(p) \in \Omega_{\phi_i}$ for every $1 \leq i \leq n$. So there exists V_{ϕ_i} an open neighbourhood of $\phi_i(p)$ in Ω_{ϕ_i} for every i. Let V_p be a neighbourhood of p such that

$$\bar{V}_p \subset \bigcap_{i=1}^{n} \phi_i^{-1}(V_{\phi_i}).$$

Let $\{\psi_n\}$ be a sequence in G and without loss of generality it can be assumed that there exists a subsequence such that $\psi_n = f_n \circ \phi_1$. Now $\phi_1(\bar{V}_p)$ is a compact subset in Ω_1 and f_n has a subsequence which either converges uniformly on $\phi_1(\bar{V}_p)$ or diverges to infinity. Thus V_p is contained in the Fatou set of G which is a contradiction! □

The next observation is an extension of the fact that if $\phi \in \mathcal{P}_k$, then $F(\phi) = F(\phi^n)$ for every $n > 0$ for the case of semigroups.

Definition 2.4. Let G be a semigroup generated by endomorphisms of $\mathbb{C}^k$. A sub semigroup H of G is said to have finite index if there is a finite collection of elements say $\psi_1, \psi_2, \ldots, \psi_{m-1} \in G$ such that

$$G = \Big(\bigcup_{i=1}^{m-1} \psi_i \circ H \Big) \cup H.$$

The index of H in G is the smallest possible number m.

Definition 2.5. A sub semigroup H of a semigroup G of endomorphisms of $\mathbb{C}^k$ is of co–finite index if there is a finite collection of elements say $\psi_1, \psi_2, \ldots, \psi_{m-1} \in G$ such that either

$$\psi \circ \psi_j \in H \text{ or } \psi \in H$$

for every $\psi \in G$ and for some $1 \leq j \leq m-1$. The index of H in G is the smallest possible number m.

Proposition 2.6. *Let G be a semigroup generated by proper holomorphic endomorphisms of $\mathbb{C}^k$ and H be a sub semigroup of G which has a finite (or co–finite) index in G. Then $F(G) = F(H)$ and $J(G) = J(H)$.*

Proof. From the definition itself it follows that $F(G) \subset F(H)$. To prove the other inclusion, pick any sequence $\{\phi_n\} \in G$. Since H has a finite index in G, there exists ψ_i, $1 \leq i \leq m-1$ such that

$$G = \Big(\bigcup_{i=1}^{m-1} \psi_i \circ H \Big) \cup H.$$

So without loss of generality one can assume that there exists a subsequence say ϕ_{n_k} with the property

$$\phi_{n_k} = \psi_1 \circ h_{n_k}$$

where $\{h_{n_k}\}$ is a sequence in H. Now on $F(H)$, the sequence $\{h_{n_k}\}$ has a convergent subsequence. Hence, so do $\{\phi_{n_k}\}$ and $\{\phi_n\}$ as ψ_1 is a proper map in $\mathbb{C}^k$. □

Let G be a semigroup

$$G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$$

where $\phi_i \in \mathcal{P}_k$, for every $1 \leq i \leq m$ and each of these ϕ_i's commute with each other, i.e., $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for $i \neq j$. Let H be a sub semigroup of G defined as

$$H = \langle \phi_1^{l_1}, \phi_2^{l_2}, \ldots, \phi_m^{l_m} \rangle$$

where $l_i > 0$ for every $1 \leq i \leq m$. Then H has a finite index in G and hence by Proposition 2.6 $F(G) = F(H)$.

Corollary 2.7. *Let ϕ_i be elements in $\mathcal{P}_k$ for $1 \leq i \leq m$, $l = (l_1, l_2, \ldots, l_m)$ a $m-$tuple of positive integers and $G_l = \langle \phi_1^{l_1}, \phi_2^{l_2}, \ldots, \phi_m^{l_m} \rangle$. Then $F(G_l)$ and $J(G_l)$ are independent of the $m-$tuple l, if $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for every $1 \leq i, j \leq m$, i.e., given two $m-$tuples p and q, $F(G_p) = F(G_q)$.*

Proof. Since G_l has a finite index in G for every $m-$tuple $l = (l_1, l_2, \ldots, l_m)$, it follows that $F(G_l) = F(G)$ and $J(G_l) = J(G)$. □

Example 2.8. Let $G = \langle f, g \rangle$ where $f(z_1, z_2) = (z_1^2, z_2^2)$ and $g(z_1, z_2) = (z_1^2/a, z_2^2)$ where $a \in \mathbb{C}$ such that $|a| > 1$. Then it is easy to check that

$$J(f) = \big\{|z_1| = 1\big\} \times \big\{|z_2| \leq 1\big\} \cup \big\{|z_1| \leq 1\big\} \times \big\{|z_2| = 1\big\}$$

and

$$J(g) = \big\{|z_1| = |a|\big\} \times \big\{|z_2| \leq 1\big\} \cup \big\{|z_1| \leq |a|\big\} \times \big\{|z_2| = 1\big\}.$$

Now consider the bidisc $\{|z_1| < 1, |z_2| < 1\}$. Clearly this domain is forward invariant under both f and g. This shows that $\{|z_1| < 1, |z_2| < 1\} \subset F(G)$. Similarly observe that

$$\{|z_2| > 1\} \cup \{|z_1| > |a|\} \subset F(G).$$

We claim that

$$\big\{1 \leq |z_1| \leq |a|\big\} \times \big\{|z_2| \leq 1\big\} \subset J(G).$$

Note that $\{|z_1| = |a|, |z_2| \leq 1\}$ is contained inside $J(G)$ and since $J(G)$ is backward invariant it follows that

$$\{|z_1| = |a|^{1/2}, |z_2| \leq 1\} \subset f^{-1}(\{|z_1| = |a|, |z_2| \leq 1\}) \subset J(G).$$

So inductively we get that

$$\{|z_1| = |a|^t, |z_2| \leq 1\} \subset J(G)$$

for any $t = k2^{-n}$ where $1 \leq k \leq 2^n$ and $n \geq 1$. As $\{k2^{-n} : 1 \leq k \leq 2^n,\ n \geq 1\}$ is dense in $[0, 1]$, it follows that $\{1 \leq |z_1| \leq |a|\} \times \{|z_2| \leq 1\} \subset J(G)$. Thus the Julia set of the semigroup G is not forward invariant and clearly from the above observations one can prove that

$$J(G) = \big\{|z_1| \leq 1\big\} \times \big\{|z_2| = 1\big\} \cup \big\{1 \leq |z_1| \leq |a|\big\} \times \big\{|z_2| \leq 1\big\}.$$

Example 2.9. Let $T_0(z) = 1,\ T_1(z) = z$ and $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$ for $n \geq 1$ and $G = \langle f_0, f_1, f_2, \ldots\rangle$, with $f_i(z_1, z_2) = (T_i(z_1), z_2^2)$ for $i \geq 0$. Consider

$$G_1 = \langle T_0(z_1), T_1(z_1), T_2(z_1), \ldots\rangle,\ G_2 = \langle z_2^2\rangle.$$

Since any sequence in G_1 is uniformly unbounded on the complement of $[-1, 1]$ it follows that

$$J(G) = [-1, 1] \times \{|z_2| \leq 1\}.$$

Also as $J(G_1) \subset \mathbb{C}$ is completely invariant so is $J(G)$.

3. Isolated points in the Julia set of a semigroup G.

Proposition 3.1. *Let $G = \langle \phi_1, \phi_2, \ldots\rangle$ where each $\phi_i \in \mathcal{E}_k$. If the Julia set $J(G)$ contains an isolated point (say a) then there exists a neighbourhood Ω_a of a such that $\Omega_a \setminus \{a\} \subset F(G)$ and $\psi \in G$ which satisfies $\Omega_a \subset\subset \psi(\Omega_a)$. In particular, if G is a semigroup generated by proper maps, then $\psi^{-1}(a) = a$.*

Proof. Assume $a = 0$ is an isolated point in the Julia set $J(G)$. Then there exists a sufficiently small ball $B(0, \epsilon)$ around 0 such that $B(0, \epsilon) \setminus \{0\}$ is contained $F(G)$. Let

$$A := \{z : \epsilon/2 \leq |z| \leq \epsilon\}.$$

Then $A \subset F(G)$.

Claim: There exists a sequence $\phi_n \in G$ such that ϕ_n diverges to infinity on A.

Suppose not. Then for every sequence $\{\phi_n\} \in G$, there exists a subsequence $\{\phi_{n_k}\}$ which converges to a finite limit in A. By the maximum modulus principle

$$\|\phi_{n_k}\|_{B(0,\epsilon)} < M.$$

By the Arzelá–Ascoli Theorem it follows that ϕ_{n_k} is equicontinuous on $B(0, \epsilon)$, which contradicts that $0 \in J(G)$.

By the same reasoning as above there exists a sequence $\{\phi_n\} \in G$ such that it diverges uniformly to infinity on A but does not diverge uniformly to infinity on $B(0, \epsilon)$, since it would again imply that $B(0, \epsilon)$ is contained in the Fatou set of G. Thus there exists a sequence of points x_n in $B(0, \epsilon)$ such that $\phi_n(x_n)$ is bounded i.e.,

$$|\phi_n(x_n)| < M$$

for some large $M > 0$. So we can choose a subsequence of this $\{\phi_n\}$ and relabel it as $\{\phi_n\}$ again such that it satisfies the following condition:

$$\phi_n(x_n) \to q \text{ and } x_n \to p$$

where $p \in \overline{B(0,\epsilon)}$.

Claim: $p = 0$.

Suppose not. Then $\phi_n(p)$ is bounded. Let $\widetilde{A} = \{z : \min(|p|, \epsilon/2) \le |z| \le \epsilon\}$. Then $\widetilde{A} \supseteq A$. Now $\phi_{n_k}(p)$ converges on $\widetilde{A}$, then ϕ_{n_k} on $\widetilde{A}$ converges to a finite limit, and hence on A by the maximum modulus principle. This is a contradiction!

Since $\phi_n|_{\partial B(0,\epsilon)} \to \infty$ for large n

$$\|\phi_n\|_{\partial B(0,\epsilon)} \gg |q|.$$

Thus for a sufficiently large $R > 0$ and n

$$B(0, |q| + R) \cap \phi_n(B(0,\epsilon)) \ne \emptyset.$$

Now, if $B(0,\epsilon) \nsubseteq \phi_n(B(0,\epsilon))$, then $B(0,|q|+R) \nsubseteq \phi_n(B(0,\epsilon))$ since $B(0,\epsilon) \subset B(0,|q|+R)$ for large $R > 0$. Then there exists $y_n \in \partial B(0,\epsilon)$ such that $|\phi_n(y_n)| < |q| + R$, which is not possible. Hence $B(0,\epsilon) \subset\subset \phi_n(B(0,\epsilon))$ for sufficiently large n. Relabel this ϕ_n as ψ and consider the neighbourhood Ω_0 as $B(0,\epsilon)$.

Since $0 \in B(0,\epsilon) \subset \psi(B(0,\epsilon))$, there exists $\alpha \in B(0,\epsilon)$ such that $\psi(\alpha) = 0$. From Proposition 2.1 it follows that $\alpha = 0$. □

Theorem 3.2. *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{I}_k$. If the Julia set $J(G)$ contains an isolated point, say a then there exists an element $\psi \in G$ such that ψ is conjugate to an upper triangular automorphism.*

Proof. Without loss of generality we can assume that $a = 0$. Now by Proposition 3.1 it follows that there exists a sufficiently small ball $B(0,\epsilon)$ around 0 and an element $\psi \in G$ such that $B(0,\epsilon) \subset\subset \psi(B(0,\epsilon))$. Since ψ is injective map in $\mathbb{C}^k$, $\psi(B(0,\epsilon))$ is biholomorphic to $B(0,\epsilon)$ and hence we can consider the inverse i.e.,

$$\psi^{-1} : \psi(B(0,\epsilon)) \to B(0,\epsilon).$$

Note that $\psi(B(0,\epsilon))$ is bounded and $B(0,\epsilon)$ is compactly contained in $\psi(B(0,\epsilon))$. Therefore there exists an $\alpha > 1$ such that the map defined by

$$\psi_\alpha = \alpha\psi^{-1}(z)$$

is a self map of the bounded domain $\psi(B(0,\epsilon))$ with a fixed point at 0. Then by the Carathéodory–Cartan–Kaup–Wu Theorem (See Theorem 11.3.1 in [3]) it follows that all the eigenvalues of ψ_α are contained in the unit disc. Hence 0 is a repelling fixed point for ψ and also is an isolated point in the Julia set of ψ.

Since $B(0,\epsilon) \setminus \{0\} \in J(G)$, $B(0,\epsilon) \setminus \{0\}$ is also contained in the Fatou set of ψ and using the same argument as in the Proposition 3.1 there exists a subsequence (say n_k) such that

$$\|\psi^{n_k}\|_{\partial B(0,\epsilon)} \to \infty$$

uniformly. Thus for any given $R > 0$ there exists k_0 large enough such that $B(0,R) \subset \psi^{n_{k_0}}(B(0,\epsilon))$. Hence ψ is an automorphism of $\mathbb{C}^k$ and the basin of attraction of ψ^{-1} at 0 is all of $\mathbb{C}^k$. Now by the result of Rosay–Rudin ([1]) ψ is conjugate to an upper triangular map. □

Remark 3.3. The proof here shows that there exists a sequence $\phi_n \in G$ such that each ϕ_n is conjugate to an upper triangular map.

Recall that a domain ω is holomorphically homotopic to a point in a domain Ω if there exists a continuous map $h : [0,1] \times \bar{\omega} \to \Omega$ with $h(1,z) = z$ and $h(0,z) = p$ where $p \in \omega$ and $h(t, \cdot)$ is holomorphic in ω for every $t \in [0,1]$.

Proposition 3.4. *Let ϕ be a non-constant endomorphism of $\mathbb{C}^k$ such that on a bounded domain $U \subset F(\phi)$, the map ϕ is proper onto its image, $U \subset\subset \phi(U)$ and U is holomorphically homotopic to a point in $\phi(U)$ then*

(i) *ϕ has a fixed point, say p in U.*
(ii) *ϕ is invertible at its fixed points.*
(iii) *The backward orbit of ϕ at the fixed point in U is finite i.e, $O^-(p) \cap U$ is finite where*

$$O^-_\phi(p) = \{z \in \mathbb{C}^k : \phi^n(z) = p, n \geq 1\}.$$

Proof. That the map ϕ has a fixed point p in U follows from Lemma 4.3 in [8].
Without loss of generality we can assume $p = 0$. Consider $\psi(z) = \phi(p+z) - p$ and $\Omega = \{z - p : z \in U\}$. Then ψ is the required map with the properties $\Omega \subset\subset \psi(\Omega)$ and 0 is a fixed point for ψ.
Suppose ψ is not invertible at 0, i.e., $A = D\psi(0)$ has a zero eigenvalue. Let λ_i, $1 \leq i \leq k$ be the eigenvalues of A. Therefore there exist an α such that $0 < \alpha < 1$ and $1 < m \leq k$ such that $0 = |\lambda_i| < \alpha$ for $1 \leq i \leq m$ and $|\lambda_i| > \alpha$ for $m < i \leq k$. Choose $\delta > 0$ such that

$$0 < \|D_{\mathbb{C}}\psi(z) - A\| < \epsilon_0 = \min\Big\{\alpha, \big||\lambda_i| - \alpha\big|\Big\}$$

for $z \in B(0,\delta)$ and $m < i \leq k$. Let Ψ be a Lipschitz map in $\mathbb{C}^k$ such that

$$Lip(\Psi) = \|A\| + \epsilon_0$$

and

$$\Psi \equiv \psi \ \text{ on } \ B(0,\delta).$$

Now

$$W_s^\Psi := \{z \in \mathbb{C}^k : |\alpha^n \Psi^n(z)| \text{ is bounded }\}$$

can be realized as a graph of a continuous function (See [2]) $G_\Psi : \mathbb{C}^m \to \mathbb{C}^{k-m}$ such that $G_\Psi(0) = 0$. Since

$$W_s^\Psi = W_s^\psi \text{ on } B(0,\delta/2)$$

$W_s^\psi \cap \Omega$ is an infinite non-empty set containing 0. Also ${\psi^{n_k}}_{|\bar{\Omega}} \to \psi_0$ for some sequence n_k and ψ_0 is holomorphic on the component (say F_0) of $F(\psi)$ containing Ω. Let

$$W_1^\psi = \{z \in F_0 : \psi^{n_k}(z) \to 0 \text{ as } k \to \infty\}.$$

Then $W_s^\psi \cap F_0 \subset W_1^\psi$ and

$$W_1^\psi = \bigcap_{i=1}^{k} {\psi_{0,i}}^{-1}(0)$$

where $\psi_{0,i}$ is the $i-$th coordinate function of ψ_0. If $W_1^\psi \cap \partial\Omega = \emptyset$ then $W_1^\psi \cap \Omega$ and hence $W_s^\psi \cap \Omega$ will have to be finite which is not true. Thus there exists a positive integer n_0 such that $\psi^{n_0}(\partial\Omega) \cap \Omega \neq \emptyset$ but by assumption it follows that $\Omega \subset\subset \psi^n(\Omega)$ for all $n \geq 1$, i.e., $\psi^n(\partial\Omega) \cap \Omega = \emptyset$ for all $n > 0$. This proves that A has no zero eigenvalues.
Note that this observation also reveals that $W_1^\psi \cap \Omega$ has to be a finite set, and since

$$O^-_\psi(0) \subset W_1^\psi$$

the backward orbit of 0 under ψ is finite. $\square$

Now we can state and prove Theorem 3.2 for semigroups generated by the elements of $\mathcal{E}_k$.

Theorem 3.5. *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. If the Julia set $J(G)$ contains an isolated point (say a) then there exists a $\psi \in G$ such that ψ is conjugate to an upper triangular automorphism.*

Proof. Assume $a = 0$. Then as before by Proposition 3.1 there exists a map $\psi \in G$ and a domain Ω such that $\Omega \subset\subset \psi(\Omega)$.

If 0 is in the Julia set of ψ then 0 is an isolated point in $J(\psi)$ and by applying Theorem 4.2 in [8], it follows that ψ is conjugate to an upper triangular automorphism.

Suppose $\Omega \subset F(\psi)$. By Proposition 3.4, ψ has a fixed point in Ω i.e.,$\{\psi^n\}$ has a convergent subsequence in $\bar{\Omega}$.

Case 1: Suppose that $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{P}_k$.

Applying Proposition 3.1, we have that $\psi^{-1}(0) = 0$ and there exists $\psi \in G$ such that

$$\Omega \subset\subset B(0, R) \subset\subset \psi(\Omega) \tag{3.1}$$

where Ω is a sufficiently small ball at 0 and $R > 0$ is a sufficiently large number. Now let ω is the component of $\psi^{-1}(B(0, R))$ in Ω containing the origin. Also from Proposition 3.4 it follows that 0 is a regular point of ψ, which implies that ψ is a biholomorphism on ω. Define Ψ_β on $\psi(\omega)$ as

$$\Psi_\beta(z) = \beta\psi^{-1}(z)$$

and note that Ψ_β is a self map of $B(0, R)$ for some $\beta > 1$ with a fixed point at 0. Then the eigenvalues of $D_{\mathbb{C}}\Psi_\beta(0)$ are in the closed unit disc, i.e.,

$$\beta|\lambda_i^{-1}| \leq 1$$

where λ_i are eigenvalues of A. Hence 0 is a repelling fixed point for the map ψ and $0 \notin F(\psi)$. Since 0 is an isolated point in the Julia set of ψ, by Theorem 4.2 in [8] ψ is conjugate to an upper triangular automorphism of $\mathbb{C}^k$.

Case 2: Suppose that $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$.

As before by Proposition 3.1 there exists $\psi \in G$ such that

$$\Omega \subset B(0, R) \subset \psi(\Omega)$$

and let ω be a component of $\psi^{-1}(B(0, R)) \subset \Omega$. Then ω satisfies all the condition of Proposition 3.4 and hence there exists a fixed point p of ψ in ω and $O_\psi^-(p) \cap \omega$ is finite.

Claim: $\psi^{-1}(p) \cap \omega = p$

Suppose not i.e.,

$$\#\{\psi^{-1}(p)\} = \text{the cardinality of } \psi^{-1}(p) = m$$

and $m \geq 2$. Let $a_1 \in \psi^{-1}(p) \setminus \{p\}$ in ω and define

$$S_1 = O_\psi^-(a_1) \cap \omega.$$

Then $S_1 \subset O_\psi^-(p) \cap \omega$. Now choose inductively $a_n \in \psi^{-1}(a_{n-1}) \setminus \{a_{n-1}\}$ for $n \geq 2$ and define

$$S_n = O_\psi^-(a_n) \cap \omega.$$

Then

$$S_n \subset S_{n-1} \text{ and } \bigcup_{i=1}^{n} S_i \subset O_\psi^-(p) \cap \omega$$

for every $n \geq 2$. Note that $a_n \notin S_n$, otherwise there is a positive integer $k_n > 0$ such that $\psi^{k_n}(a_n) = a_n$ i.e., a_n is a periodic point of ψ, and

$$\psi^{k_n+m}(a_n) = p$$

for any $m > n$. Since $O^-_\psi(p) \cap \omega$ is finite it follows that S_n has to be empty for large n. This implies that there exists a $n_0 \geq 1$ such that $\psi^{-1}(a_{n_0}) = a_{n_0}$ and $a_{n_0} \in \omega$. But by Proposition 3.4 ψ is invertible at its fixed points which means that a_{n_0} is a regular value of ψ and

$$\#\{\psi^{-1}(a_{n_0})\} = m \geq 2$$

which is a contradiction! Hence the claim.
Now by similar arguments as in the case of proper maps it follows that ψ is a biholomorphism from ω to $B(0, R)$ and p is a repelling fixed point of ψ and hence lies in $J(\psi) \subset J(G)$. Since $\omega \cap J(G) = \{0\}$, we have $p = 0$ which is an isolated point in the Julia set of ψ and hence ψ is conjugate to an upper triangular automorphism. $\square$

4. Recurrent and Wandering Fatou components of a semigroup G.

As discussed in Section 1 we will be studying the properties of recurrent and wandering Fatou components of semigroup generated by entire maps of maximal generic rank on $\mathbb{C}^k$. The wandering and the recurrent Fatou components for a semigroup G are defined as:

Definition 4.1. Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. Given a Fatou component Ω of G and $\phi \in G$, let Ω_ϕ be the Fatou component of G containing $\phi(\Omega \setminus \Sigma_\phi)$ where Σ_ϕ is the set where the Jacobian of ϕ vanishes. A Fatou component is *wandering* if the set $\big\{\Omega_\phi : \phi \in G\big\}$ contains infinitely many distinct elements.

Definition 4.2. Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. A Fatou component Ω of G is *recurrent* if for any sequence $\{g_j\}_{j \geq 1} \subset G$, there exists a subsequence $\{g_{j_m}\}$ and a point $p \in \Omega$ (the point p depends on the chosen sequence) such that $g_{j_m}(p) \to p_0 \in \Omega$.

Note that we assume here a stronger definition of recurrence than the existing definition for the case of iterations of a single holomorphic endomorphism of $\mathbb{C}^k$. The natural extension of this definition to the semigroup set up would have been the following, a Fatou component Ω is recurrent if there is a point $p \in \Omega$ and a sequence $\phi_n \in \Omega$ such that $\phi_n(p) \to p_0$, where $p_0 \in \Omega$. If this definition of recurrence is adopted then it is possible that a *Recurrent* domain is *Wandering.* In particular, Theorem 5.3 in [11] gives an example of a polynomial semigroup $G = \langle \phi_1, \phi_2, \ldots \rangle$ in $\mathbb{C}$, such that there exists a Fatou component, (say $\mathcal{B}$, which is conformally equivalent to a disc), that is wandering, but returns to the same component infinitely often. This means that there exists sequences say $\phi_n^+ \in G$ and $\phi_n^- \in G$ such that $\phi_n^-(\mathcal{B}) \subset \mathcal{B}$ or $\phi_n^+(\mathcal{B})$ are contained in distinct Fatou components of G. This example can be easily adapted in higher dimensions.

Example 4.3. Consider the semigroup $\mathcal{G} = \langle \Phi_1, \Phi_2, \ldots, \rangle$ generated by the maps

$$\Phi_i(z, w) = (\phi_i(z), w^2)$$

where ϕ_i are the polynomial maps as in Theorem 5.3 of [11]. Let $\{\Phi_n^-\}_{n \geq 1} \subset G$ be the sequence that maps $\mathcal{B} \times \mathbb{D}$ into itself and $\{\Phi_n^+\}_{n \geq 1} \subset G$ be the sequence such that

$$\Phi_i^+(\mathcal{B} \times \mathbb{D}) \cap \Phi_j^+(\mathcal{B} \times \mathbb{D}) = \emptyset$$

for every $i \neq j$. Also $\mathcal{B} \times \mathbb{D}$ is a Fatou component of $\mathcal{G}$ as any point on the boundary of $\mathcal{B} \times \mathbb{D}$, is either in the Julia set of G or in the Julia set of the map $z \to z^2$. Hence $\mathcal{B} \times \mathbb{D}$ is a Fatou component which is wandering, but may be recurring as well if we adapt the classical definition of recurrence.

Hence we work with a stronger definition of recurrence than the classical one. Next we provide an alternative description for recurrent Fatou components of G.

Lemma 4.4. *A Fatou component Ω is recurrent if and only if for any sequence $\{\phi_j\} \subset G$, there exists a compact set $K \subset \Omega$ and a subsequence $\{\phi_{j_m}\}$ such that $\phi_{j_m}(p_{j_m}) \to p_0 \in \Omega$ for a sequence $\{p_{j_m}\} \subset K$.*

Proof. Take any sequence $\{\phi_j\} \subset G$. Then there exists a subsequence $\{\phi_{j_m}\}$ and points $\{p_{j_m}\} \subset K$ with K compact in Ω such that

$$\phi_{j_m}(p_{j_m}) \to p_0 \in \Omega.$$

Without loss of generality we assume $p_{j_m} \to q_0 \in K$. It follows that $\phi_{j_m}(q_0) \to p_0 \in \Omega$ using the fact that any sequence of G is normal on the Fatou set of G. $\square$

Proposition 4.5. *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. If Ω is a recurrent Fatou component of G, then G is locally bounded on Ω. Moreover Ω is pseudoconvex and Runge.*

Proof. Assume G is not locally bounded on Ω. Then there exists a compact set $K \subset \Omega$ and $\{g_r\} \subseteq G$ such that $|g_r(z_r)| > r$ with $z_r \in K$ for every $r \geq 1$. Clearly this can not be the case since Ω is a recurrent Fatou component, so we can always get a subsequence $\{g_{r_k}\}$ from the sequence $\{g_r\} \in G$ such that it converges to a holomorphic function uniformly on compact set in Ω and in particular on K. From the proof of Proposition 2.2, it follows that local boundedness of G on Ω implies that Ω is polynomially convex. Hence Ω is pseudoconvex. $\square$

Theorem 4.6. *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. Assume that Ω is a recurrent Fatou component of G. If there exists a $\phi \in G$ such that $\phi(\Omega)$ is contained in the Fatou set of G i.e., $\phi(\Omega) \subset F(G)$ then one of the following is true*

(i) *There exists an attracting fixed point (say p_0) in Ω for the map ϕ.*
(ii) *There exists a closed connected submanifold $M_\phi \subset \Omega$ of dimension r_ϕ with $1 \leq r_\phi \leq k-1$ and an integer $l_\phi > 0$ such that*
  (a) *ϕ^{l_ϕ} is an automorphism of M_ϕ and $\overline{\{\phi^{nl_\phi}\}_{n\geq 1}}$ is a compact subgroup of* $\mathrm{Aut}(M_\phi)$.
  (b) *If $f \in \overline{\{\phi^n\}}$, then f has maximal generic rank r_ϕ in Ω.*
(iii) *ϕ is an automorphism of Ω and $\overline{\{\phi^n\}}$ is a compact subgroup of* $\mathrm{Aut}(\Omega)$.

Proof. Since $\Omega \subset F(G)$, there exists a recurrent Fatou component of the map ϕ (say Ω_ϕ) such that $\Omega \subset \Omega_\phi$, i.e., there exists an integer $l \geq 1$ such that

$$\phi^l(\Omega_\phi) \cap \Omega_\phi \neq \emptyset \ \text{ and } \ \phi^m(\Omega_\phi) \cap \Omega_\phi = \emptyset$$

for $0 \leq m < l$. So, if $l > 1$ then there do not exist any $p \in \Omega$ such that any subsequence of $\{\phi^{lk+1}(p)\}_{k\geq 1}$ converges to a point in Ω. Hence $l = 1$ and by assumption it follows that $\phi(\Omega) \subset \Omega$.

Let h be a limit function of $\{\phi^n\}$ of maximal rank (say r_ϕ). i.e.,

$$h(p) = \lim_{j\to\infty} \phi^{n_j}(p) \ \text{ for every } \ p \in \Omega,$$

where $\{n_j\}$ is an increasing subsequence of natural numbers.

Case 1: If $r_\phi = 0$. Then $h(\Omega) = p_0$ for some $p_0 \in \Omega$ since by recurrence there exists a point $p \in \Omega$, such that $\phi^{n_j}(p) \to p_0$ and $p_0 \in \Omega$. Also $h(p_0) = p_0$. Then

$$\phi(p_0) = \phi(h(p_0)) = h(\phi(p_0)) = p_0,$$

i.e., p_0 is a fixed point of ϕ. As some sequence of iterates of ϕ converge to a constant function, p_0 is an attracting fixed point for ϕ.

Case 2: If $r_\phi \geq 1$. Then there exists an increasing subsequence $\{m_j\}$ such that

$$p_j = m_{j+1} - m_j$$

are increasing positive integers and the sequences $\{\phi^{m_j}\}$ and $\{\phi^{p_j}\}$ converge uniformly to the limit functions h and $\tilde{h}$ respectively on the Fatou component Ω. Since by recurrence $h(\Omega)\cap\Omega \neq \emptyset$, if $p \in \Omega$ be such that $p = h(q)$ for some $q \in \Omega$ then

$$\tilde{h}(p) = \lim_{j\to\infty} \phi^{m_{j+1}-m_j}(p) = \lim_{j\to\infty} \phi^{m_{j+1}-m_j}(\phi^{m_j}(q)) = p$$

Define

$$M = \{x \in \Omega : \tilde{h}(x) = x\}.$$

Claim: M is a closed complex submanifold of Ω.

Since $h(\Omega)\cap\Omega \subset M$, M is a variety of dimension $\geq r_\phi$. But by the choice of h, the generic rank of $\tilde{h} \leq r_\phi$ and $M \subset \tilde{h}(\Omega)\cap\Omega$. So the dimension of M is r_ϕ. Now for any point in M, the rank of the derivative matrix of $\mathrm{Id} - \tilde{h}$ is greater than or equal to $k - r_\phi$. Suppose for some $x \in M$ the rank of $D(\mathrm{Id} - \tilde{h})(x) > k - r_\phi$, then there exists a small neighbourhood of x, say V_x such that $V_x \subset \Omega$ and

$$\text{rank of } \mathrm{Id} - \tilde{h} > k - r_\phi \ \text{ for every } \ x \in V_x.$$

Then $\{\mathrm{Id} - \tilde{h}\}^{-1}(0) \cap V_x$ is a variety of dimension at most $r_\phi - 1$ i.e., the dimension of M is strictly less than r_ϕ, which is a contradiction. Thus the rank of $\mathrm{Id} - \tilde{h}$ is $k - r_\phi$ for every point in M and hence M is a closed submanifold of Ω.

Step 1: Suppose that $r_\phi = k$.

Then clearly $M = \Omega$ and $\tilde{h}$ on Ω is the identity map. Let $h_2 = \lim \phi^{p_j - 1}$. Then

$$\tilde{h}(x) = h_2 \circ \phi(x) = x, \ \text{ for every } \ x \in \Omega$$

i.e., ϕ is injective on Ω and $\phi(\Omega)$ is an open subset of Ω. Suppose there exists an $x \in \Omega \setminus \phi(\Omega)$ then for a sufficiently small ball of radius $r > 0$ with $B_r(x) \subset \Omega$

$$\phi^l(\Omega) \cap B_r(x) = \emptyset \ \text{ for every } \ l \geq 1.$$

This contradicts that $\phi^{p_j}(x) \to x$. Hence ϕ is surjective on Ω and hence an automorphism of Ω.

Step 2: Suppose that $1 \leq r_\phi \leq k-1$. Let M_ϕ denote an irreducible component of M. For every $q \in M_\phi$, it follows that $\phi^{p_j}(q) \to q$ as $j \to \infty$. Since $\phi(\Omega) \subset \Omega$, we get $\phi^n(q) \in \Omega$ for every $n \geq 1$ and

$$\tilde{h} \circ \phi^n(q) = \phi^n \circ \tilde{h}(q) = \phi^n(q) \text{ for every } q \in M_\phi,$$

i.e., $\phi^n(M_\phi) \subset M$ for every $n \geq 1$.

Claim: There exists a positive integer l_ϕ such that $\phi^{l_\phi}(M_\phi) \subset M_\phi$.

Let $p_0 \in M_\phi$ and $\Delta \subset \Omega$ be a polydisk at p_0 such that Δ does not intersect the other components of M_ϕ. Now choose $\Delta' \subset \Delta$, a sufficiently small polydisk such that $\tilde{h}(\Delta') \subset \Delta$. Then $\omega = \tilde{h}(\Delta') \subset M_\phi$ is a $r_\phi-$ dimensional manifold. Let Δ'' be a $r_\phi-$ dimensional polydisk inside ω and $\{w_l\}_{l\geq 1}$ be a sequence in Δ'' such that it converges to some $w_0 \in \Delta''$. But $\phi^{p_j}(w_{p_j}) \to w_0$ as $j \to \infty$ hence

$$\phi^{p_j}(M_\phi) \cap \Delta \neq \emptyset, \text{ i.e., } \phi^{p_j}(M_\phi) \subset (M_\phi)$$

for j sufficiently large. Let l_ϕ be the minimum value such that M_ϕ is invariant under ϕ^{l_ϕ}.

Claim: ϕ^{l_ϕ} is an automorphism of M_ϕ.

Without loss of generality there exists a sequence $\{k_j\}$ such that $p_j = i_0 + k_j l_\phi$ for some $0 \leq i_0 \leq l_\phi - 1$ i.e.,

$$\phi^{i_0} \circ \phi^{k_j l_\phi}(x) \to x \ \text{ for every } \ x \in M_\phi.$$

As M_ϕ is invariant under ϕ^{l_ϕ}, the sequence $x_j = \phi^{k_j l_\phi}(x)$ lies in M_ϕ. Again as before let Δ_x be a sufficiently small neighbourhood such that $\Delta_x \subset \Omega$ and Δ_x does not intersect the other

components of M. Since $\phi^{i_0}(x_j) \in \Delta_x \cap M_\phi$ for large j, $\phi^{i_0}(M_\phi) \subset M_\phi$. But $0 \leq i_0 \leq l_\phi - 1$, i.e., $i_0 = 0$ and $\{\phi^{k_j l_\phi}\}$ converges uniformly to the identity on M_ϕ. Let $\psi = \lim \phi^{(k_j-1)l_\phi}$ then

$$\phi^{l_\phi} \circ \psi(x) = \psi \circ \phi^{l_\phi}(x) = x \text{ for every } x \in M_\phi.$$

Hence ϕ^{l_ϕ} is injective on M_ϕ and $\phi^{l_\phi}(M_\phi)$ is an open subset in the manifold M_ϕ. Now as in *Step 1* observe that $\phi^{k_j l_\phi}$ converges to the identity on M_ϕ for an unbounded sequence $\{k_j\}$, so ϕ^{l_ϕ} is also surjective on M_ϕ. Thus the claim.

Let $Y = \{\phi^{n l_\phi}\}_{n \geq 1} \subset \mathrm{Aut}(M_\phi)$.

Claim: $\bar{Y}$ is a locally compact subgroup of $\mathrm{Aut}(M_\phi)$.

For some $\Psi \in Y$ and for a compact set $K \subset M_\phi$ consider the neighbourhood of Ψ given by

$$V_\Psi(K, \epsilon) = \{\psi \in \mathrm{Aut}(M_\phi) : \|\psi(z) - \Psi(z)\|_K < \epsilon\}.$$

One can choose ϵ and K sufficiently small such that for every sequence $\psi_j \in V_\Psi(K, \epsilon)$ there exists an open set $U \subset \Omega$ such that $\psi_j(U \cap M_\phi) \subset \bar{V} \cap M_\phi \subset \Omega$, where V is some open subset of Ω.

Since $\psi_j = \phi^{n_j l_\phi}$ for a sequence $\{n_k\}$ and Ω is a Fatou component, ψ_j has a convergent subsequence in Ω. We choose appropriate subsequences such that the limit maps

$$\Psi_1 = \lim_{j \to \infty} \phi^{n_j l_\phi} \text{ and } \Psi_2 = \lim_{j \to \infty} \phi^{(k_j - n_j) l_\phi}$$

is defined on Ω. Also as M_ϕ is closed in Ω, $\Psi_i(M_\phi) \subset \overline{M_\phi}$ for every $i = 1, 2$ where $\overline{M_\phi}$ denote the closure of M_ϕ in $\mathbb{C}^k$. Then $\Psi_1(U) \subset \Omega$ and

$$\Psi_2 \circ \Psi_1(x) = x \text{ for every } x \in U \cap M_\phi. \tag{4.1}$$

Since Ψ_1 on M_ϕ is a limit of automorphisms of M_ϕ, the Jacobian of Ψ_1 on the manifold M_ϕ is either non-zero at every point of M_ϕ or vanishes identically. But by (4.1), Ψ_1 restricted to $U \cap M_\phi$ is injective, which is open in the manifold M_ϕ i.e., Ψ_1 is an open map of M_ϕ and $\Psi_1(M_\phi) \subset M_\phi$. So (4.1) is true for every $x \in M_\phi$. Now by the same arguments it follows that Ψ_2 is an injective map from M_ϕ such that $\Psi_2(M_\phi) \subset M_\phi$. Hence

$$\Psi_2 \circ \Psi_1(x) = \Psi_1 \circ \Psi_2(x) = x \text{ for every } x \in M_\phi,$$

i.e. Ψ_1 is an automorphism of M_ϕ. This proves that $\bar{Y}$ is a locally compact subgroup of $\mathrm{Aut}(M_\phi)$.

Now since M_ϕ is a complex manifold and $\bar{Y}$ is a locally abelian subgroup of automorphisms of M_ϕ, by Theorem A in [6], it follows that $\bar{Y}$ is a Lie group. Hence the component of $\bar{Y}$ containing the identity is isomorphic to $\mathbb{T}^l \times \mathbb{R}^m$. Suppose Ψ is the isomorphism, then for some $n > 0$, $\Psi(a, b) = \phi^{n l_\phi}$. Now if $b \neq 0$, then there does not exist an increasing sequence of k_j such that $\phi^{k_j l_\phi}$ converges to identity. This proves that the component of $\bar{Y}$ containing the identity is compact and hence any component of $\bar{Y}$ is compact by the same arguments. Also as M_ϕ is contained in the Fatou set, the number of components of $\bar{Y}$ is finite, thus $\bar{Y}$ is a compact subgroup of $\mathrm{Aut}(M_\phi)$.

If $r_\phi = k$, then M_ϕ is Ω, then one can apply the same technique as discussed above to conclude that $\overline{\{\phi^n\}}$ is a closed compact subgroup of $\mathrm{Aut}(\Omega)$.

Finally, let f be a limit of $\{\phi^n\}_{n \geq 1}$ i.e.,

$$f(p) = \lim_{j \to \infty} \phi^{n_j}(p) \text{ for every } p \in \Omega.$$

Claim: The generic rank of f is r_ϕ.

By the definition of recurrence it follows that $\Omega \subset \Omega_\phi$, where Ω_ϕ is a periodic Fatou component for ϕ with period 1. Hence by Theorem 3.3 in [8] it follows that the limit maps of the set $\{\phi^n\}$

in Ω_ϕ have the same generic rank (say r). But Ω is an open subset of the Fatou component Ω_ϕ, so the rank of limit maps restricted to Ω should be same, i.e., $r = r_\phi$ and each limit map of $\{\phi^n\}$ has rank r_ϕ. $\square$

By Proposition 4.5 a semigroup G is always locally uniformly bounded on a recurrent Fatou component semigroup G. If G is finitely generated by holomorphic endomorphisms of maximal rank k in $\mathbb{C}^k$, then by Proposition 2.2 it follows that a recurrent Fatou component is mapped in the Fatou set by any elemnet of G. Hence we have the following corollary.

Corollary 4.7. *Let $G = \langle \phi_1, \phi_2, \dots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. Assume that Ω is a recurrent Fatou component of G then for every $\phi \in G$ one of the following is true*

(i) *There exists an attracting fixed point (say p_0) in Ω for the map ϕ.*
(ii) *There exists a closed connected submanifold $M_\phi \subset \Omega$ of dimension r_ϕ with $1 \leq r_\phi \leq k-1$ and an integer $l_\phi > 0$ such that*
  (a) *ϕ^{l_ϕ} is an automorphism of M_ϕ and $\overline{\{\phi^{nl_\phi}\}_{n\geq 1}}$ is a compact subgroup of* $\mathrm{Aut}(M_\phi)$.
  (b) *If $f \in \overline{\{\phi^n\}}$, then f has maximal generic rank r_ϕ in Ω.*
(iii) *ϕ is an automorphism of Ω and $\overline{\{\phi^n\}}$ is a compact subgroup of* $\mathrm{Aut}(\Omega)$.

Example 4.8. Let $G = \langle \phi_1, \phi_2 \rangle$ be a semigroup of entire maps in $\mathbb{C}^2$ generated by

$$\phi_1(z, w) = (w, \alpha z - w^2) \text{ and } \phi_2(z, w) = (zw, w)$$

where $0 < \alpha < 1$. Then G is locally uniformly bounded on a sufficiently small neighbourhood around the origin, and $\phi(0) = 0$ for every $\phi \in G$. So the Fatou component of G containing 0 (say Ω_0) is recurrent. Now note that for ϕ_2

$$r_{\phi_2} = 1 \quad \text{and} \quad M_{\phi_2} = \{(0, w) : w \in \mathbb{C}\} \cap \Omega_0,$$

whereas for ϕ_1 the origin is an attracting fixed point. This illustrates the different behaviour of the sequences $\{\phi_1^n\}$ and $\{\phi_2^n\}$ (both of which are in G) on Ω_0.

Note that for any other $\phi \in G$ which is not of the form $\phi_1^k,\ k \geq 2$, contains a factor of ϕ_2 at least once. Since for a small enough ball (say B) around origin, ϕ_2 is contracting, and $\phi_1(B) \subset B$ so there exists a constant $0 < a_\phi < 1$ such that

$$|\phi(z)| \leq a_\phi |z| \text{ for every } z \in B,$$

i.e., the origin is an attracting fixed point.

Proposition 4.9. *Let $G = \langle \phi_1, \phi_2, \dots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let Ω be an invariant Fatou component of G. Then either Ω is recurrent or there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.*

Proof. If Ω is not recurrent, then there exists a sequence $\{\phi_n\} \subset G$ such that $\{\phi_n\} \to \partial\Omega \cup \{\infty\}$ uniformly on compact sets of Ω. Assume $\{\phi_{n_k}\}$ converges to a holomorphic function f on Ω . This implies that $f(\Omega) \subset \partial\Omega$ contradicting the assumption that each ϕ_{n_k} is volume preserving. Hence $\{\phi_{n_k}\}$ diverges to infinity uniformly on compact subsets of Ω. $\square$

Proposition 4.10. *Let $G = \langle \phi_1, \phi_2, \dots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let Ω be a wandering Fatou component of G. Then there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.*

Proof. Since Ω is wandering, one can choose a sequence $\{\phi_n\} \subset G$ so that

$$\Omega_{\phi_n} \cap \Omega_{\phi_m} = \emptyset \tag{4.2}$$

for $n \neq m$. If this sequence $\{\phi_n\}$ does not diverge to infinity uniformly on compact subsets, some subsequence $\{\phi_{n_k}\}$ will converge to a holomorphic function h on Ω. By abuse of notation, we denote $\{\phi_{n_k}\}$ still by $\{\phi_n\}$. Fix $z_0 \in \Omega$. Then for any given ϵ, there exists δ such that

$$|\phi_{n_0}(z) - \phi_n(z)| < \epsilon \tag{4.3}$$

for all $n \geq n_0$ and for all $z \in B(z_0, \delta)$. From (4.3) it follows that $\mathrm{vol}(\cup_{n \geq n_o} \phi_n(B(z_0, \delta)))$ is finite. On the other hand, since each ϕ_n is volume preserving and (4.2) holds, we get

$$\mathrm{Vol}\Big(\bigcup_{n \geq n_o} \phi_n\big(B(z_0, \delta)\big)\Big) = +\infty.$$

Hence we have proved the existence of a sequence in G converging to infinity. $\square$

5. Concluding Remarks

As mentioned in the introduction, the classification of recurrent Fatou components for iterations of holomorphic endomorphisms of complex projective spaces has been studied in [9] and [7]. It would be interesting to explore the same question for semigroups of holomorphic endomorphisms of complex projective spaces. The main theorem in [9] and [7] are proved under the assumption that the given recurrent Fatou component is also forward invariant. The analogue of such a condition in the case of semigroups is not clear to us since we are then dealing with a family of maps none of which is distinguishable from the other.

References

1. Jean-Pierre Rosay and Walter Rudin, *Holomorphic maps from* $\mathbf{C}^n$ *to* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86. MR 929658 (89d:32058)
2. Jean-Christophe Yoccoz, *Introduction to hyperbolic dynamics*, Real and complex dynamical systems (Hillerød, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 464, Kluwer Acad. Publ., Dordrecht, 1995, pp. 265–291. MR 1351526 (96h:58131)
3. Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1992 edition. MR 1846625 (2002e:32001)
4. Eric Bedford and John Smillie, *Polynomial diffeomorphisms of* $\mathbf{C}^2$*. II. Stable manifolds and recurrence*, J. Amer. Math. Soc. **4** (1991), no. 4, 657–679. MR 1115786 (92m:32048)
5. Mikhail Lyubich and Han Peters, *Classification of invariant Fatou components for dissipative Hénon maps*, Geom. Funct. Anal. **24** (2014), no. 3, 887–915. MR 3213832
6. Salomon Bochner and Deane Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. (2) **47** (1946), 639–653. MR 0018187 (8,253c)
7. John Erik Fornæss and Feng Rong, *Classification of recurrent domains for holomorphic maps on complex projective spaces*, J. Geom. Anal. **24** (2014), no. 2, 779–785. MR 3192297
8. John Erik Fornæss and Nessim Sibony, *Fatou and Julia sets for entire mappings in* $\mathbf{C}^k$, Math. Ann. **311** (1998), no. 1, 27–40. MR 1624255 (99e:32044)
9. ———, *Classification of recurrent domains for some holomorphic maps*, Math. Ann. **301** (1995), no. 4, 813–820. MR 1326769 (96c:32030)
10. Brendan J. Weickert, *Nonwandering, nonrecurrent Fatou components in* $\mathbf{P}^2$, Pacific J. Math. **211** (2003), no. 2, 391–397. MR 2015743 (2004m:37090)
11. A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions. I*, Proc. London Math. Soc. (3) **73** (1996), no. 2, 358–384. MR 1397693 (97e:58198)

Sayani Bera: Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
E-mail address: `sayani@math.iisc.ernet.in`

Ratna Pal: Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
E-mail address: `ratna10@math.iisc.ernet.in`